Pairs of Rank and Kernel Dimension for $\mathbb{Z}_2\mathbb{Z}_4$-linear Codes

C. Fernández-Córdoba
Universitat Autònoma de Barcelona
cristina.fernandez@autonoma.edu

(joint work with J. Pujol and M. Villanueva)

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-*additive code*, which is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. The code $\mathcal{C}$ is isomorphic to $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$. Let $\mathcal{C}_b$ be the subcode of $\mathcal{C}$ which contains all order two codewords and κ the dimension of the punctured code of $\mathcal{C}_b$ by deleting the $\mathbb{Z}_4$ coordinates. The $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is of type $(\alpha, \beta; \gamma, \delta; \kappa)$, the length is $\alpha + \beta$ and the number of codewords is $2^{\gamma+2\delta}$.

We will take an extension $\Phi : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_2^n$, for $n = \alpha + 2\beta$, of the usual Gray map, $\phi : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2^2$ where $\phi(0) = (0,0)$, $\phi(1) = (0,1)$, $\phi(2) = (1,1)$ and $\phi(3) = (1,0)$, given by $\Phi(x,y) = (x, \phi(y_1), \ldots, \phi(y_\beta))$, for any $x \in \mathbb{Z}_2^\alpha$ and any $y = (y_1, \ldots, y_\beta) \in \mathbb{Z}_4^\beta$. This Gray map is an isometry which transforms Lee distances defined in the $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{C}$ over $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ to Hamming distances defined in the binary codes $C = \Phi(\mathcal{C})$. If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, the binary image $C = \Phi(\mathcal{C})$ is a $\mathbb{Z}_2\mathbb{Z}_4$-*linear code* of length $n = \alpha + 2\beta$ and type $(\alpha, \beta; \gamma, \delta; \kappa)$.

The rank, kernel and dimension of the kernel are defined for binary codes and they are specially useful for binary non-linear codes. The *rank* of a binary code C, $r = rank(C)$, is simply the dimension of $\langle C \rangle$, which is the linear span of the codewords of C. The *kernel* of a binary code C, $K(C)$, is the set of vectors that leave C invariant under translation, i.e. $K(C) = \{x \in \mathbb{Z}_2^n \mid C + x = C\}$. If C contains the all-zero vector, then $K(C)$ is a binary linear subcode of C. We show that for binary codes which are $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, we can also define the kernel using the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. In this case, in order to compute the kernel $K(C)$ of a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C is much easier if we consider the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C} = \Phi^{-1}(C)$ and we compute $\mathcal{K}(\mathcal{C}) = \Phi^{-1}(K(C))$ using a generator matrix of $\mathcal{C}$. We also prove that if C is a $\mathbb{Z}_2\mathbb{Z}_4$-linear code, then $K(C)$ and $\langle C \rangle$ are also $\mathbb{Z}_2\mathbb{Z}_4$-linear codes. Moreover, since $K(C) \subseteq C \subseteq \langle C \rangle$ and C can be written as the union of cosets of $K(C)$, we also have that, equivalently, $\mathcal{K}(\mathcal{C}) \subseteq \mathcal{C} \subseteq \mathcal{S}_\mathcal{C}$, where $\mathcal{S}_\mathcal{C} = \Phi^{-1}(\langle C \rangle)$, and $\mathcal{C}$ can be written as cosets of $\mathcal{K}(\mathcal{C})$.

Using combinatorial enumeration techniques, we establish lower and upper bounds for the possible values of these parameters. We also give the construction of a $\mathbb{Z}_2\mathbb{Z}_4$-linear code with rank r (resp. kernel dimension k) for each feasible value r (resp. k). Finally, we establish the bounds on the rank, once the dimension of the kernel is fixed, and we give the construction of a $\mathbb{Z}_2\mathbb{Z}_4$-linear code with rank r and kernel dimension k for each possible pair (r, k).

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